AD-A257 237

Projection with a Minimal System of Inequalities

by

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92-28280

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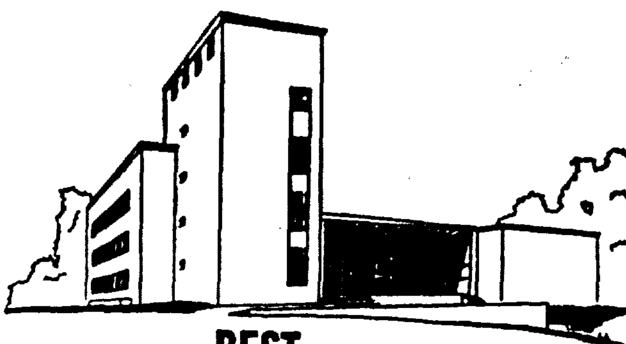

Management Science Research Report #MSRR-585

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[1]The research underlying this report was supported by the National Science Foundation, Grant #DDM-8901495 and the Office of Naval Research through Contract N00014-85-K-0198.

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Abstract

Projection of a polyhedron involves the use of a cone whose extreme rays induce the inequalities defining the projection. These inequalities need not be facet defining.

We introduce a transformation that produces a cone whose extreme rays induce facets of the projection.

Given a polyhedron

$$Q := \{(u,x) \in \mathcal{R}^p \times \mathcal{R}^q : Au + Bx \leq d\},$$

where A, B and d have m rows, the *projection* of Q onto the subspace of the x-variables is

$$P_x(Q) := \{x \in \mathcal{R}^q : \exists u \in \mathcal{R}^p \text{ with } (u,x) \in Q\}.$$

We are discussing the projection of a polyhedron, but the projection of a more general set can often be reduced to that of a polyhedron: for instance, if instead of Q we consider the nonpolyhedral set

$$S := \{(u,x) \in \mathcal{R}^p \times \mathcal{R}^q : Au + Bx \leq d, \ x \in X\}$$

where X is arbitrary, then

$$P_x(S) = P_x(Q) \cap X.$$

It is well known (see, for instance, [2], Section IV, or [5], Chapter I.4.4) that

$$P_x(Q) = \{x \in \mathcal{R}^q : (vB)x \leq vd \text{ for all } v \in extr\ W\}$$

where *extr* W denotes the set of extreme rays of the *projection cone*

$$W := \{v \in \mathcal{R}^m : vA = 0, \ v \geq 0\}.$$

It is also well known that although the inequalities defining $P_x(Q)$ are in 1-1 correspondence with the extreme rays of W, they do not necessarily define facets of $P_x(Q)$, i.e. the system defining $P_x(Q)$ is not necessarily minimal. As pointed out by M. Goemans in his recent talk at IPCO II [4], in practice often large numbers of redundant inequalities are generated, even though only extreme rays of W are used. It would be nice to be able to tell which extreme rays of W induce facets of $P_x(Q)$, but that question has no answer in terms of the properties of W only: whether the inequality $(vB)x \leq vd$ defines a facet of $P_x(Q)$ depends on B and d as well as on v itself. Instead of trying to answer that question, therefore, we address a related problem: is there an alternative representation of Q, whose projection cone (or its projection on a subspace) has the property that *all* of its extreme rays induce facets of the projection? In the sequel we give such a representation. Its key feature, which brings about the desired property, is that in this representation the coefficient matrix of x is the identity matrix plus possibly some zero rows, while the right hand side is the unit vector with 1 in the last position.

Let $\text{rank}(B) = r$, and w.l.o.g. assume that the first r rows and columns of B are linearly independent. (Recall that B is $m \times q$). Perform the following sequence of operations on the system defining Q:

1. If $r = m$, let $B_1 := B$ and go to 2. Otherwise add to B $m - r$ new columns $b^{q+1}, \ldots, b^{q+m-r} \in \mathcal{R}^m$, where for $j = 1, \ldots, m-r$, $b^{q+j} := e_{r+j}$, and where e_ℓ is the unit vector in $\mathcal{R}^m$ with 1 in position ℓ. Call the resulting matrix B_1 and go to 2.

2. If $r = q$, i.e. if B_1 is $m \times m$ with $\text{rank}(B_1) = m$, let $B_0 := B_1$, $A_0 := A$, $d_0 = d$, and go to 3. Otherwise add to B_1 $q - r$ new rows $b_{m+1}, \ldots, b_{m+q-r} \in \mathcal{R}^{m+q-r}$, where for $i = 1, \ldots, q-r$, $b_{m+i} = e^T_{r+i}$, and where e^T_ℓ is the transpose of the unit vector in $\mathcal{R}^{m+q-r}$ with 1 in position $r+i$. Call the resulting matrix B_0. Add $q-r$ zero rows to A and call the resulting matrix A_0. Finally, add to d $q-r$ components equal to M (a number sufficiently large for the resulting inequality to be redundant), and call the resulting vector d_0. Then go to 3.

3. Let B_0 be $n \times n$. By construction, B_0 is nonsingular. Consider now the polyhedron

$$Q' := \left\{ (u, s, x') \in \mathcal{R}^p \times \mathcal{R}^n \times \mathcal{R}^n \;\middle|\; \begin{array}{c} A_0 u + Is + B_0 x' = d_0, \; s \geq 0 \\ x'_{q+j} = 0 \text{ for } j = 1, \ldots, n-q \end{array} \right\}$$

$$= \left\{ (u, s, x') \in \mathcal{R}^p \times \mathcal{R}^n \times \mathcal{R}^n \;\middle|\; \begin{array}{c} B_0^{-1} A_0 u + B_0^{-1} s + x' = B_0^{-1} d_0, \; s \geq 0 \\ x'_{q+j} = 0 \text{ for } j = 1, \ldots, n-q \end{array} \right\}$$

and rewrite Q' as

$$Q'' = \left\{ (u, s, u_0, x') \in \mathcal{R}^p \times \mathcal{R}^n \times \mathcal{R} \times \mathcal{R}^n \;\middle|\; \begin{array}{r} B_0^{-1} A_0 u + B_0^{-1} s - B_0^{-1} d_0 u_0 + x' = 0, \; s \geq 0 \\ u_0 = 1 \\ x'_{q+j} = 0 \text{ for } j = 1, \ldots, n-q \end{array} \right\}$$

4. Let $B_0^{-1} = \begin{pmatrix} B_{0q}^{-1} \\ B_{0,n-q}^{-1} \end{pmatrix}$, where B_{0q}^{-1} and $B_{0,n-q}^{-1}$ are the submatrices containing the first

q and the last $n-q$ rows of B_0^{-1}, respectively, and let

$$Q^0 := \left\{ (u,s,u_0,x) \in \mathcal{R}^p \times \mathcal{R}^n \times \mathcal{R} \times \mathcal{R}^q \;\middle|\; \begin{array}{lllllll} B_{0q}^{-1} A_0 u & + & B_{0q}^{-1} s & - & B_{0q}^{-1} d_0 u_0 & + \; x & = \; 0 \\ B_{0,n-q}^{-1} A_0 u & + & B_{0,n-q}^{-1} s & - & B_{0,n-q}^{-1} d_0 u_0 & & = \; 0 \\ & & & & u_0 & & = \; 1 \\ & & s & & & & \geq \; 0 \end{array} \right\}.$$

5. To project Q^0 onto the x-space, define the projection cone

$$W^0 := \left\{ (v,w,v_0) \in \mathcal{R}^q \times \mathcal{R}^{n-q} \times \mathcal{R} \;\middle|\; \begin{array}{llllll} vB_{0q}^{-1} A_0 & + & wB_{0,n-q}^{-1} A_0 & & & = \; 0 \\ v(-B_{0q}^{-1}) A_0 & + & w(-B_{0,n-q}^{-1}) A_0 & + & v_0 & = \; 0 \\ vB_{0q}^{-1} & + & wB_{0,n-q}^{-1} & & & \geq \; 0 \end{array} \right\}.$$

Note that, since B_0^{-1} is nonsingular, W^0 is pointed. The projection of Q^0 is then

$$P_x(Q^0) = \left\{ x \in \mathcal{R}^q \;\middle|\; \begin{array}{c} vx \leq v_0 \text{ for all } (v,v_0) \in \mathcal{R}^q \times \mathcal{R} \text{ such that} \\ (v,w,v_0) \in \mathit{extr}\ W^0 \text{ for some } w \in \mathcal{R}^{n-q} \end{array} \right\},$$

where $\mathit{extr}\ W^0$ denotes the set of extreme rays of W^0.

It is easy to see that $P_x(Q^0) \equiv P_x(Q)$, since there is a 1-1 correspondence between the points of Q and those of Q^0.

Consider now the polyhedral cone polar to $P_x(Q^0)$, namely

$$P_x^*(Q^0) = \left\{ (v,v_0) \in \mathcal{R}^q \times \mathcal{R} \,\middle|\, vx \leq v_0 \text{ for all } x \in P_x(Q^0) \right\}.$$

By construction,

$$\begin{aligned} P_x^*(Q^0) &= \{(v,v_0) \in \mathcal{R}^q \times \mathcal{R} : (v,w,v_0) \in W^0 \text{ for some } w \in \mathcal{R}^{n-q}\}, \\ &\equiv P_{(v,v_0)}(W^0), \end{aligned}$$

i.e. the polar cone of the projection of Q^0 on the x-space is the projection of W^0 on the (v,v_0)-space.

But then from basic properties of polarity, we have the following.

Theorem 1 *Let $P_x(Q)$ be full dimensional. Then the inequality $vx \leq v_0$ defines a facet of $P_x(Q)$ if and only if (v,v_0) is an extreme ray of the cone $P_{(v,v_0)}(W^0)$.*

Proof. If $P_x(Q)$ is full dimensional, so is $P_x(Q^0)$; and from basic properties of polarity, there is a 1-1 correspondence between facets of the polyhedron $P_x(Q^0)$ and extreme rays of its polar cone $P_x^*(Q^0)$. But $P_x^*(Q^0) \equiv P_{(v,v_0)}(W^0)$.□

Note that if $(\bar{v}, \bar{v}_0)$ is an extreme ray of $P_{(v,v_0)}(W^0)$, then there exists $\bar{w} \in \mathcal{R}^{n-q}$ such that $(\bar{v}, \bar{w}, \bar{v}_0)$ is an extreme ray of W^0. The converse, however, is not always true.

On the other hand, in the important special case when the matrix B is of full row rank, i.e. when $r = m$ and $n = q$, we have

Corollary 2 *Let $P_x(Q)$ be full dimensional, and rank$(B) = m$. Then the inequality $vx \leq v_0$ defines a facet of $P_x(Q)$ if and only if (v, v_0) is an extreme ray of the cone W^0.*

A few comments are in order.

First, note that the definition of Q does not contain explicitly constraints of the form $u \geq 0$; if they are present, they are part of the system $Au + Bx \leq d$, i.e. A and B are of the form $A = \begin{pmatrix} A' \\ -I \end{pmatrix}$ and $B = \begin{pmatrix} B' \\ 0 \end{pmatrix}$, respectively, where I is the $p \times p$ identity matrix and 0 is the $p \times q$ zero matrix.

Second, while the above described transformation produces a projection cone with a very desirable property, there is a price to pay for this: if the matrix A has a structure that makes it easy to generate the extreme rays of W, that structure gets lost in the transformation, and the extreme rays of W^0, or $P_{(v,v_0)}(W^0)$, may be much harder to generate.

Third, in many important cases the matrix B is of the form $B = \begin{pmatrix} I \\ 0 \end{pmatrix}$, which voids the need for the above transformation and produces directly a projection cone $W := \{(v, w, v_0) : (v, w)A = 0,\ (v, w) \geq 0\}$ such that the extreme rays of $P_{(v,v_0)}(W)$, the projection of W onto the (v, v_0)-space, yield facets of $P_x(Q)$. This is the case encountered, for instance, in the characterization of the perfectly matchable subgraph polytope of a graph in [3]; as well as in the recent lift-and-project cutting plane procedure of [1]. In these cases, it is important to know how to use the cone W to generate extreme rays of $P_{(v,v_0)}(W)$ (see [1] and its references for a discussion of this issue, which was investigated in the 70's).

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REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER MSRR-585	2. GOVT ACCESSION NO
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle) PROJECTION WITH A MINIMAL SYSTEM OF INEQUALITIES	5. TYPE OF REPORT & PERIOD COVERED Technical Report: July 1992
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(S) Egon Balas	8. CONTRACT OR GRANT NUMBER(S) DDM-8901495 N00014-85-K-0198
9. PERFORMING ORGANIZATION NAME AND ADDRESS Graduate School of Industrial Administration Carnegie Mellon University Pittsburgh, PA 15213-3890	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Personnel and Training Research Programs Office of Naval Research (Code 434) Arlington, VA 22217	12. REPORT DATE July 1992
	13. NUMBER OF PAGES 5
14. MONITORING AGENCY NAME & ADDRESS (If different from Controlling Office)	15. SECURITY CLASS (of this report)
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Projection
Polyhedral Combinatorics

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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